Mathematics: The core course for A-level; Bostock and Chandler

Chapter 7: Trigonometric Identities

Exercise 7d

1. (a) $2\sin 14^\circ \cos 14^\circ = \sin(2\times 14) = \sin 28^\circ$

(b) $\dfrac{2\tan 35^\circ}{1-\tan^2 35^\circ} = \tan(2\times 35) = \tan 70^\circ$

(c) $1 - 2\sin^2 4\theta = \cos(2\times 4\theta) = \cos 8\theta$

(d) $\dfrac{2\tan 3\theta}{1-\tan^2 3\theta} = \tan(2\times 3\theta) = \tan 6\theta$

(e) $\sqrt{1+\cos 6\theta}$: use $2\cos^2 A = 1+\cos 2A$

$\therefore \cos A = \sqrt{\dfrac{1+\cos 2A}{2}}$

i.e. $\sqrt{2}\cos A = \sqrt{1+\cos 2A}$

Hence $\sqrt{1+\cos 6\theta} = \sqrt{2}\cos 3\theta$

(f) $\cos^2 26^\circ - \sin^2 26^\circ = \cos(2 \times 26) = \cos 52^\circ$

(g) $\sin\theta \cos\theta = \frac{1}{2}(2\sin\theta\cos\theta) = \frac{1}{2}\sin 2\theta$

(h) $2\cos^2 34^\circ - 1 = \cos(2 \times 34) = \cos 68^\circ$

(i) $\dfrac{1+\tan x}{1-\tan x} = \dfrac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x}$

$= \tan(45^\circ + x)$

(3) $\tan\theta = -\frac{7}{24}$ implies that θ lies in quadrant II or IV.

But θ is obtuse $(90^\circ \le \theta \le 180^\circ)$, so $\tan\theta = -\frac{7}{24}$ lies in quadrant II.

$\therefore$ $\frac{\theta}{2}$ lies in quadrant I & $\tan\frac{\theta}{2}$ is $\therefore$ positive

$$\tan\theta = \tan\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \frac{2\tan\theta/2}{1-\tan^2\theta/2}$$

$$= \frac{2t}{1-t^2} \quad ; \quad t = \tan\frac{\theta}{2}$$

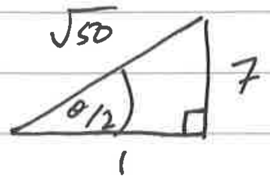
So $-\frac{7}{24} = \frac{2t}{1-t^2} \Rightarrow 7t^2 - 48t - 7 = 0$

$\therefore (t-7)(7t+1) = 0$

So $t = 7$ or $-\frac{1}{7}$

$\therefore$ $\tan\frac{\theta}{2} = 7$ or $\tan\frac{\theta}{2} = -\frac{1}{7}$

But $\tan\frac{\theta}{2}$ is Positive $\therefore$ $\tan\frac{\theta}{2} = 7$



So $\sin\frac{\theta}{2} = \frac{7}{\sqrt{50}} = \frac{7}{\sqrt{50}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{7\sqrt{2}}{\sqrt{100}} = \frac{7}{10}\sqrt{2}$$

Similarly $\cos\frac{\theta}{2} = \frac{1}{\sqrt{50}} = \frac{1}{10}\sqrt{2}$.

$$\sin\theta = \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \sin 2\left(\frac{\theta}{2}\right)$$

$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\left(\frac{7}{10}\sqrt{2}\right)\cdot\left(\frac{1}{10}\sqrt{2}\right)$$

$$= \frac{7}{25}$$

$$\cos\theta = \cos 2\left(\frac{\theta}{2}\right) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$= \left(\frac{1}{10}\sqrt{2}\right)^2 - \left(\frac{7}{10}\sqrt{2}\right)^2$$

$$= -\frac{24}{25}$$

$\tan\theta = -\frac{7}{24}$, $\sin\theta = \frac{7}{25}$ $\cos\theta = -\frac{24}{25}$

as θ is obtuse (as stated in the question)
$\Rightarrow$ θ lies in quadrant II

$\therefore$ These three trig ratios are consistent.

(4) (a) $x = \tan 2\theta \quad , \quad y = \tan\theta$

Now, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \quad x = \dfrac{2y}{1-y^2} \qquad$ So $\quad 2y = x(1-y^2)$

(b) $x = \cos 2\theta \quad , \quad y = \cos\theta$

Now, $\cos 2\theta = 2\cos^2\theta - 1$

So $\quad x = 2y^2 - 1$

(c) $x = \cos 2\theta \quad , \quad y = \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$$\frac{1}{y} = \sin\theta$$

Now, $\cos 2\theta = 1 - 2\sin^2\theta$

So $\quad x = 1 - 2\left(\dfrac{1}{y}\right)^2$

$1 - x = 2\left(\dfrac{1}{y^2}\right) \quad \Rightarrow \quad y^2(1-x) = 2$

(d) $x = \sin 2\theta$, $y = \sec 4\theta = \dfrac{1}{\cos 4\theta}$

So $\dfrac{1}{y} = \cos 4\theta$

Now, if $\cos 2\theta = 1 - 2\sin^2\theta$
Then $\cos 4\theta = 1 - 2\sin^2 2\theta$

So $\dfrac{1}{y} = 1 - 2x^2 \Rightarrow 1 = y(1 - 2x^2)$

(5) $\dfrac{1 - \cos 2A}{\sin 2A} \equiv \tan A$

LHS: $1 - \cos 2A \equiv 1 - (1 - 2\sin^2 A)$

$\sin 2A \equiv 2 \sin A \cos A$

So $\dfrac{1 - \cos 2A}{\sin 2A} \equiv \dfrac{1 - (1 - 2\sin^2 A)}{2 \sin A \cos A} \equiv \dfrac{\sin A}{\cos A} \equiv \tan A$ ✓

(6) $\tan\theta + \cot\theta \equiv 2 \operatorname{cosec} 2\theta$

LHS: $\tan\theta + \cot\theta \equiv \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

$\equiv \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta \cos\theta}$

So $\quad \tan\theta + \cot\theta \equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta$ ✓

(7) $\sec 2A + \tan 2A \equiv \dfrac{\cos A + \sin A}{\cos A - \sin A}$

LHS: $\dfrac{1}{\cos 2A} + \dfrac{\sin 2A}{\cos 2A} \equiv \dfrac{1 + \sin 2A}{\cos 2A}$

$$\equiv \frac{1 + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Now, $\sin^2 A + \cos^2 A = 1$ and (by $a^2 - b^2 = (a-b)(a+b)$) we have
$\cos^2 A - \sin^2 A = (\cos A - \sin A)(\cos A + \sin A)$

So $\quad \sec 2A + \tan 2A \equiv \dfrac{\sin^2 A + \cos^2 A + 2\sin A\cos A}{(\cos A - \sin A)(\cos A + \sin A)}$

$$\equiv \frac{(\sin A + \cos A)^2}{(\cos A - \sin A)(\cos A + \sin A)}$$

$$\equiv \frac{\cos A + \sin A}{\cos A - \sin A}$$

(8) $\dfrac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} \equiv \tan A$

LHS: $\quad 1 - \cos 2A + \sin 2A \equiv 1 - (\cos^2 A - \sin^2 A) + 2\sin A\cos A$

$$\equiv 1 - \cos^2 A + \sin^2 A + 2\sin A\cos A$$

$$\equiv 2\sin^2 A + 2\sin A\cos A$$

$$1 + \cos 2A + \sin 2A \equiv 1 + (2\cos^2 A - 1) + 2\sin A \cos A$$

$$\equiv 2\cos^2 A + 2\sin A \cos A$$

$$\therefore \quad LHS \equiv \frac{2\sin^2 A + 2\sin A \cos A}{2\cos^2 A + 2\sin A \cos A}$$

$$\equiv \frac{\sin A \, (\sin A + \cos A)}{\cos A \, (\sin A + \cos A)}$$

$$\equiv \tan A \checkmark$$

(9) $\cos 4A \equiv 8\cos^4 A - 8\cos^2 A + 1$

LHS: $\cos 4A \equiv \cos(2A + 2A) \equiv \cos 2A \cos 2A - \sin 2A \sin 2A$

change everything to cosine of individual angles:
So

$$\cos 4A \equiv (2\cos^2 A - 1)^2 - (2\sin A \cos A)^2$$

$$\equiv 4\cos^4 A - 4\cos^2 A + 1 - 4\sin^2 A \cos^2 A$$

$$\equiv 4\cos^4 A - 4\cos^2 A + 1 - 4(1 - \cos^2 A)\cos^2 A$$

$$\equiv 4\cos^4 A - 4\cos^2 A + 1 - 4\cos^2 A + 4\cos^4 A$$

$$\equiv 8\cos^4 A - 8\cos^2 A + 1 \checkmark$$

(10) $\sin 2\theta \equiv \dfrac{2\tan\theta}{1+\tan^2\theta}$

The RHS looks familiar : $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$

so $\sin 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta} \cdot \cos 2\theta$

$$\underline{RHS} \equiv \frac{2\sin\theta/\cos\theta}{1-\dfrac{\sin^2\theta}{\cos^2\theta}} \cdot \left(\cos^2\theta - \sin^2\theta\right)$$

$$\equiv \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \cdot \left(\cos^2\theta - \sin^2\theta\right) \equiv \sin 2\theta \equiv LHS \checkmark$$

or Back to the original question:

$$RHS \equiv \frac{2\tan\theta}{1+\tan^2\theta} \equiv \frac{2\sin\theta/\cos\theta}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$\equiv \frac{\cos^2\theta \cdot 2\sin\theta/\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$\equiv 2\sin\theta\cos\theta \equiv \sin 2\theta \equiv LHS \checkmark$$

(11) $\cos 2\theta \equiv \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$$RHS: \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} \equiv \frac{1-\sin^2\theta/\cos^2\theta}{1+\sin^2\theta/\cos^2\theta}$$

$$\equiv \frac{\cos^2\theta\,(1 - \sin^2\theta/\cos^2\theta)}{\cos^2\theta + \sin^2\theta}$$

$$\equiv \cos^2\theta - \sin^2\theta \equiv \cos 2\theta . \checkmark$$

(12) $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$

LHS: $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$$\equiv (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$

$$\equiv 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

$$\equiv 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$$

$$\equiv 4\cos^3\theta - 3\cos\theta \quad \checkmark$$

(13) $\cos 2x = \sin x$

$1 - 2\sin^2 x = \sin x$

$\therefore \quad 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

So $\sin x = \frac{1}{2}$ or -1

For $\sin x = \frac{1}{2}$: $x = 30° \pm 360° n$, $n = 0, 1, 2, \ldots$

and $x = 150° \pm 360° n$, $n = 0, 1, 2, \ldots$

For $\sin x = -1$: $x = -90° \pm 360° n$, $n = 0, 1, 2, \ldots$

In $0° \le \theta \le 360°$: For $\sin x = \frac{1}{2}$, $x = 30°, 150°$

For $\sin x = -1$, $x = 270°$

(14) $\sin 2x + \cos x = 0$

So $2 \sin x \cos x + \cos x = 0$

$\therefore \cos x (2 \sin x + 1) = 0$

$\Rightarrow \cos x = 0$ and / or $\sin x = -\frac{1}{2}$

For $\cos x = 0$: $x = 90 \pm 180° n$, $n = 0, 1, 2, 3, \ldots$

For $\sin x = -\frac{1}{2}$: $x = -30° \pm 360° n$, $n = 0, 1, 2, 3, \ldots$

and $x = -150° \pm 360° n$, $n = 0, 1, 2, 3, \ldots$

In $0° \le \theta \le 360°$: For $\cos x = 0$, $x = 90°, 270°$

For $\sin x = -\frac{1}{2}$: $x = 210°, 330°$

(15) $4 - 5\cos\theta = 2\sin^2\theta$

$\therefore\ 4 - 5\cos\theta = 2(1 - \cos^2\theta)$

$= 2 - 2\cos^2\theta$

So $2\cos^2\theta - 5\cos\theta + 2 = 0$

$(2\cos\theta - 1)(\cos\theta - 2) = 0$

So $\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$. This last one is not valid, $\therefore\ \cos\theta = \frac{1}{2}$

Hence $\theta = 60^\circ \pm 360^\circ n$, $n = 0, 1, 2, \ldots$
and $\theta = -60^\circ \pm 360^\circ n$, $n = 0, 1, 2, \ldots$

In $0 \le \theta \le 360^\circ$: $\theta = 60^\circ, 300^\circ$

(16) $\tan\theta\tan 2\theta = 2$

So $\tan\theta \dfrac{2\tan\theta}{1 - \tan^2\theta} = 2$

$\therefore\ \tan\theta(2\tan\theta) = 2(1 - \tan^2\theta)$

$\therefore\ 4\tan^2\theta = 2 \Rightarrow \tan^2\theta = \frac{1}{2} \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{2}}$

For $\tan\theta = +\frac{1}{\sqrt{2}}$: $\theta = 35.26^\circ \pm 180^\circ n$, $n = 0, 1, 2, \ldots$

For $\tan\theta = -\frac{1}{\sqrt{2}}$: $\theta = -35.26^\circ \pm 180^\circ n$, $n = 0, 1, 2, \cdots$

For $0 \le \theta \le 360^\circ$: $\theta = 35.26^\circ$

$35.26^\circ + 180^\circ = 215.26^\circ$

and

$\theta = -35.26 + 360^\circ = 324.74^\circ$

$= -35.26 + 180^\circ = 144.74^\circ$

(17) $\sin 2\theta - 1 = \cos 2\theta$

$\therefore\ 2\sin\theta\cos\theta - 1 = 2\cos^2\theta - 1$

So $\sin\theta\cos\theta = \cos^2\theta$

$\therefore\ \sin\theta\cos\theta - \cos^2\theta = 0$

$\cos\theta(\sin\theta - \cos\theta) = 0$ (Do not cancel $\cos\theta$s. Factolise, else you lose one answer)

$\therefore\ \cos\theta = 0$ and $\sin\theta = \cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = 1$

For $\cos\theta = 0$: $\theta = 90^\circ \pm 180^\circ n$, $n = 0, 1, 2, \ldots$

For $\tan\theta = 1$: $\theta = 45^\circ \pm 180^\circ n$, $n = 0, 1, 2, \ldots$

and $\theta = -135^\circ \pm 180^\circ n$ But This is covered By The previous line

For $0 \leq \theta \leq 360^\circ$: $\theta = 90^\circ, 270^\circ$

and $\theta = 45^\circ, 225^\circ$

(18) $5\cos x \sin 2x + 4\sin^2 x = 4$

So $5\cos x\,(2\sin x\cos x) + 4\sin^2 x = 4$

$\therefore \quad 10\sin x\cos^2 x + 4\sin^2 x = 4 \qquad (*)$

$10\sin x\,(1 - \sin^2 x) + 4\sin^2 x = 4$

$10\sin x - 10\sin^3 x + 4\sin^2 x = 4$

Looks too complicated. So go back to $(*)$ and swap for $\sin^2 x$:

$\therefore \quad 10\sin x\cos^2 x + 4(1 - \cos^2 x) = 4$

$10\sin x\cos^2 x + 4 - 4\cos^2 x = 4$

$\therefore \quad \cos^2 x\,(10\sin x - 4) = 0$

$\therefore \quad \cos^2 x = 0 \Rightarrow \cos x = 0$

and/or $\quad 10\sin x - 4 = 0 \Rightarrow \sin x = \frac{2}{5}$

For $\cos x = 0$: $\quad \theta = 90 \pm 180° n \; , \quad n = 0, 1, 2, \cdots$

For $\sin x = \frac{2}{5}$: $\quad \theta = 23.578° \pm 360° n \; , \quad n = 0, 1, 2, \cdots$

and $\theta = 156.422° \pm 360° n \; , \quad n = 0, 1, 2, \cdots$

IN $0 \leq \theta \leq 360°$: $\quad \theta = 90°, 270°$

and $\theta = 23.578°, 156.422°$